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Zad.2.

a)

f - bijekcja $:\Leftrightarrow$ f -surjekcja i f - iniekcja

f - surjekcja $:\Leftrightarrow$ $\sqsupset$ f = Y $\Leftrightarrow$ $\sqsupset$ f = [-1,1]

f -iniekcja $:\Leftrightarrow \underset{x_1, x_2 \in D_f}{\forall} \quad f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

Zauważmy, że f jest nieparzysta. Zatem wystarczy rozważyć funkcje w przedziale $[0,+\infty]$

f - surjekcja $:\Leftrightarrow$ $\sqsupset$ f = [0,1]

$\sqsupset f = \{y \in Y : \underset{x \in X}{\exists} \; y = f(x)\}$

$$\sqsupset f = \left\{ \frac{x}{x+1} : \underset{x \in [0,+\infty]}{\exists} \right\} = \left[\frac{0+1}{0}, 1 \right] = [0,1]$$

zatem $\sqsupset$ f = [-1,1] $\Rightarrow$ f -suriekcja

f -iniekcja $:\Leftrightarrow \underset{x_1, x_2 \in R}{\forall} \quad f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

z założenia $\underset{x_1, x_2 \in R}{\forall} \quad f(x_1) = f(x_2) \Rightarrow f(x_1) - f(x_2) = 0 \Rightarrow \frac{x_1}{x_1+1} - \frac{x_2}{x_2+1} = 0 \Rightarrow$

$$\frac{x_1(x_2+1) - x_2(x_1+1)}{(x_1+1)(x_2+1)} = 0 \Rightarrow x_1x_2 + x_1 - x_2x_1 - x_2 = 0 \Rightarrow x_1 - x_2 = 0 \Rightarrow x_1 = x_2 \quad c.n.d.$$

f -surjekcja i f - iniekcja $\Rightarrow$ f - bijekcja

b) Uzasadnienie analogiczne jak w zadaniu 1.

c)

$$K(0,\frac{1}{2}) = \left\{ x \in \bar{R} : d(x,0) < \frac{1}{2} \right\}$$

$1^0 \quad x \in R$

$$d(x,0) = |\; f(x) - f(0) \;| = \left| \frac{x}{1+|x|} - 0 \right| = \left| \frac{x}{1+|x|} \right|$$

$$d(x,0) < \frac{1}{2} \Leftrightarrow \left| \frac{x}{1+|x|} \right| < \frac{1}{2}$$

$$\left| \frac{x}{1+|x|} \right| < \frac{1}{2}$$

$$\begin{cases} x \geq 0 \\ \left|\frac{x}{1+x}\right| < \frac{1}{2} \end{cases} \quad \vee \quad \begin{cases} x < 0 \\ \left|\frac{x}{1-x}\right| < \frac{1}{2} \end{cases}$$

$$\left|\frac{x}{1+x}\right| < \frac{1}{2} \quad \vee \quad \left|\frac{x}{1-x}\right| < \frac{1}{2}$$

$$\frac{x}{1+x} < \frac{1}{2} \quad \vee \quad \frac{x}{1-x} < \frac{1}{2} \quad \wedge \quad \frac{x}{1-x} > -\frac{1}{2}$$

$$\frac{x-1}{2(x+1)} < 0 \quad \vee \quad \frac{3x-1}{2(1-x)} < 0 \quad \wedge \quad \frac{x+1}{2(1-x)} > 0$$

$$x \in]-1,1[\qquad x \in [-\infty, \frac{1}{3}[\cup]1, \infty] \quad \wedge \quad x \in]-1,1[$$

$$\begin{cases} x \geq 0 \\ x \in]-1,1[\end{cases} \quad \vee \quad \begin{cases} x < 0 \\ x \in]-1, \frac{1}{3}[\end{cases}$$

$$x \in [0,1[\qquad x \in]-1,0[$$

$$x \in]-1,1[$$

$2^0 \quad x = +\infty$

$d(x,0) = |f(x) - f(0)| = |1-0| = 1$

$d(x,0) < \frac{1}{2} \Leftrightarrow 1 < \frac{1}{2} \Rightarrow x \in \varnothing$

$3^0 \quad x = -\infty$

$d(x,0) = |f(x) - f(0)| = |-1-0| = 1$

$d(x,0) < \frac{1}{2} \Leftrightarrow 1 < \frac{1}{2} \Rightarrow x \in \varnothing$

$K(0, \frac{1}{2}) =]-1,1[$

$$K(+\infty,\frac{3}{2})=\left\{x\in \bar{R} : d\left(x,+\infty\right)<\frac{3}{2}\right\}$$

$1^0 \quad x\in R$

$$d(x,+\infty)=\left|f(x)-f(+\infty)\right|=\left|\frac{x}{1+|x|}-1\right|$$

$$d(x,+\infty)<\frac{3}{2}\Leftrightarrow\left|\frac{x}{1+|x|}-1\right|<\frac{3}{2}$$

$$\left|\frac{x-1-|x|}{1+|x|}\right|<\frac{3}{2}$$

$$\begin{cases}x\geq 0\\ \left|\frac{x-1-x}{1+x}\right|<\frac{3}{2}\end{cases} \quad\vee\quad \begin{cases}x<0\\ \left|\frac{x-1+x}{1-x}\right|<\frac{3}{2}\end{cases}$$

$$\left|\frac{-1}{1+x}\right|<\frac{3}{2} \quad\vee\quad \left|\frac{2x-1}{1-x}\right|<\frac{3}{2}$$

$$\frac{1}{1+x}<\frac{3}{2} \quad\vee\quad \frac{2x-1}{1-x}<\frac{3}{2} \quad\wedge\quad \frac{2x-1}{1-x}>-\frac{3}{2}$$

$$\frac{-3x-1}{2(1+x)}<0 \quad\vee\quad \frac{7x-5}{2(1-x)}<0 \quad\wedge\quad \frac{x+1}{2(1-x)}>0$$

$$x\in[-\infty,-1[\cup]-\frac{1}{3},\infty]$$

$$\vee\quad x\in[-\infty,\frac{5}{7}[\cup]1,\infty] \quad\wedge\quad x\in]-1,1[$$

$$\begin{cases}x\geq 0\\ x\in[-\infty,-1[\cup]-\frac{1}{3},\infty]\end{cases} \quad\vee\quad \begin{cases}x<0\\ x\in]-1,\frac{5}{7}[\end{cases}$$

$$x\geq 0 \qquad\qquad x\in]-1,0[$$

$$x\in]-1,\infty]$$

$2^0 \quad x=-\infty$

$$d(x,+\infty)=\left|f(x)-f(\infty)\right|=\left|-2\right|=2$$

$$d(x,+\infty)<\frac{3}{2}\Leftrightarrow 2<\frac{3}{2}\Rightarrow x\in\varnothing$$

$$K(+\infty,\frac{3}{2})=]-1,\infty]$$